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A Theory for the Small Deformations of Cylindrical Shells
Part I - Rotationally Symmetric Deformations

EDWARD L. REISS

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A THEORY FOR THE SMALL DEFORMATIONS OF CYLINDRICAL SHELLS

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1. Introduction

The deformations of an elastic cylinder are determined if a solution of the equations of the exact three dimensional linear theory of elasticity, hereafter referred to as the exact theory, can be obtained which satisfies the prescribed force and displacement conditions on the surfaces and edges of the cylinder. Since it is difficult, in general, to construct this solution approximate theories have been proposed (for example, see ref. 1-5). These theories, called thin shell theories, are derived from sets of à priori assumptions concerning the distribution of stresses in the shell and consist of a determined system of differential equations, boundary conditions and stress displacement relations. These assumptions, which are often mutually contradictory and/or inconsistent with the exact theory, can be accurate only if the shell is sufficiently thin. In spite of these inadequacies reasonable approximations are obtained in many cases. However, it is difficult to ascertain the relative accuracies of these thin shell theories, which vary in complexity, and to establish their relation to the exact theory. It is not our intention to present a complete or critical review of the existing literature. For discussions of this nature the reader is directed to references 2, 5, 6, 7, 8 and references contained therein.

It is reasonably conjectured that the thin shell theories may yield inaccurate results in the neighborhood of boundaries (for example, near the edges of the cylinder or in a region adjacent to a hole in the cylinder wall) and for "thicker" shells. Attempts have been made, by employing "less restrictive" <u>à priori</u> assumptions, to obtain appropriate thick shell theories. Some of these theories[5,9,10] are, in a sense, generalizations of Reissner's theory for thick plates.[11] See ref. 8 for references and a discussion of some recent work. However, the nature of the assumptions again makes it difficult to estimate the accuracy of such theories and their relation to the exact theory.

It is the purpose of this paper to rectify, to some extent, the inadequacies in these approaches to shell theory by systematically deriving from the exact theory approximate thin and thick shell theories. This includes boundary conditions as well as differential equations and stress displacement relations. We only consider the rotationally symmetric deformations of a cylindrical shell of radius R and uniform wall thickness 2h. However the procedures that are employed may be suitably extended to unsymmetric deformations and to shells of other shapes.

The method of derivation begins by assuming, after appropriately scaling the independent and dependent variables,[†] that each stress and displacement component can be represented by a power series expansion in ε, a small parameter which is a

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† The proper scaling is a crucial point in the method. It determines the type and "width" of boundary layers under consideration.

positive power of h/R. These series are then substituted into the exact equations of elasticity and the exact boundary conditions on the inner and outer surfaces of the shell. By equating coefficients of similar powers of ε in these equations a sequence of algebraic and differential equations is obtained to determine the coefficients in the expansions of the stresses and displacements. The first of this set of differential equations and stress displacement relations are identical with those of the classical thin shell theory of Love.[1] Higher order terms yield "thick shell" corrections. However, these expansions do not satisfy the boundary conditions along the edges of the shell nor can they represent the stresses and displacements in a region, called the boundary layer, adjacent to the edge. To determine, from the exact theory, appropriate boundary conditions for the differential equations and to examine the stresses in the neighborhood of the edge we employ a boundary layer expansion technique related to the one proposed by Friedrichs[12] and later Friedrichs and Dressler[13] in a study of the bending of plates. The expansion and scaling procedures that we employ differ somewhat from those in refs. 12 and 13. In some sense our method is a generalization of the one given in these references. The stresses in the neighborhood of the edge are then systematically approximated by employing the solutions of a sequence of boundary value problems for the biharmonic equation in a semi-infinite strip. We then show, in terms of the definition given in Section 3,

that there is only one scaling of the independent variables, i.e. one boundary layer, that leads to a "reasonable shell theory".

A portion of our results are obtained in a somewhat less systematic manner, in an unpublished paper by Johnson and Reissner.[15] However the boundary layer is not analyzed in this reference: so that the boundary conditions for the differential equations are essentially assumed rather than systematically derived from the exact theory, and no procedure is given to approximate the stresses near the edge. The relation of the results of ref. 15 to those of this paper will be made clearer in the following.

2. Formulation

To study the deformations of a circular cylindrical shell it is convenient to introduce a cylindrical coordinate system (r,θ,x).[†] We consider a cylindrical shell of radius R, thickness 2h, and length L as a three dimensional body bounded by the coaxial cylindrical surfaces $r = R + h$ (called the outer surface) and $r = R - h$ (called the inner surface), and the parallel planes $x = 0$ and $x = L$. The annular regions, $x = 0, L$, $R - h \leq r \leq R + h$, are called the edges of the shell. The

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† It is customary in shell theory to introduce the symbol x instead of z to denote the axial coordinate. The symbol z is usually reserved for representing distances along the normal to the middle surface of the cylinder.

surface $r = R$ is referred to as the middle surface. We should, for consistency with the scaling procedure introduced in eqs. (1), place the origin of coordinates at the center of the cylinder with the edges then given by, $x = \pm L/2$. However, it is convenient for our purposes, and there is no loss of generality, to have the origin coincide with the end of the shell.

It is assumed that the cylinder is constructed from a homogeneous and isotropic elastic material and that the deformations are sufficiently small so that the linear theory of elasticity is valid. The external forces acting on the cylinder are normal forces on the inner and outer surfaces and normal and tangential forces applied to the edges. All forces are distributed in a rotationally symmetric manner. If it is assumed that the resulting deformations of the cylinder are also rotationally symmetric, then the only non-vanishing displacements, μ and ω in the x and r directions respectively, and the only non-vanishing stresses, τ_{rr}, $\tau_{\theta\theta}$, τ_{xx}, and τ_{rx}, are functions of x and r. We define the parameter, ε, as,

$$\varepsilon = \left(\frac{h}{R}\right)^{\frac{1}{b}} , \qquad (1)$$

where b is a positive integer and introduce the dimensionless variables:

$$\xi = \frac{x}{R\varepsilon^{a}} = \frac{x}{R\left(\frac{h}{R}\right)^{a/b}} , \qquad \zeta = \frac{r-R}{R\varepsilon^{b}} = \frac{r-R}{h} , \qquad (2a)$$

$$\left.\begin{aligned}
u(\xi,\zeta;\varepsilon) &= \frac{\mu(x,r)}{R} , & w(\xi,\zeta;\varepsilon) &= \frac{\omega(x,r)}{R} , \\
\sigma_x(\xi,\zeta;\varepsilon) &= \frac{\tau_{xx}(x,r)}{E} , & \sigma_\theta(\xi,\zeta;\varepsilon) &= \frac{\tau_{\theta\theta}(x,r)}{E} , \\
\sigma_z(\xi,\zeta;\varepsilon) &= \frac{\tau_{rr}(x,r)}{E} , & \sigma_{zx}(\xi,\zeta;\varepsilon) &= \frac{\tau_{rx}(x,r)}{E} .
\end{aligned}\right\} \qquad (2b)$$

Here E is Young's modulus and a is a non-negative integer such that,†

$$\lambda = b - a > 0 . \qquad (3)$$

With eq. (3) the possible (a,b) combinations are limited to the sector, $\pi/4 < \alpha \leq \pi/2$, in the a,b plane where α is the polar angle.

The independent variables, ξ and ζ, are respectively the scaled axial coordinate and the scaled "thickness" coordinate. From (2a) the equations for the outer and inner surfaces are given respectively by,†† $\zeta = +1$, $\zeta = -1$. An interpretation of the scaling procedure is given in Section 4.

In terms of the dimensionless variables (2) the equations of the exact theory with rotational symmetry are written as[1]:

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† In the following section it is shown that values of a and b are determined by the order of magnitude of the external forces. The inequality in (3) is a consequence of the scaling (2a) and the "stretching" procedures introduced in Section 4.

†† It is possible to obtain a more general scaling procedure than (1) and (2a) by replacing the exponent, 1/b, in (1) by an arbitrary non-negative integer. The complexity of the calculations are then considerably increased and no new information is obtained.

Equilibrium Equations,

$$\frac{\partial\sigma_z}{\partial\zeta} + \varepsilon^{\lambda}\frac{\partial\sigma_{zx}}{\partial\xi} + g(\zeta)(\sigma_z - \sigma_\theta) = 0, \quad \frac{\partial\sigma_{zx}}{\partial\zeta} + \varepsilon^{\lambda}\frac{\partial\sigma_x}{\partial\xi} + g(\zeta)\sigma_{zx} = 0 \ ; \quad (4)$$

Stress-Displacement Relations (Hooke's Law),

$$\left.\begin{aligned} &\frac{\partial u}{\partial\xi} = \varepsilon^{a}[\sigma_x - \nu(\sigma_\theta + \sigma_z)] \ , \qquad && w = (1+\varepsilon^{b}\zeta)[\sigma_\theta - \nu(\sigma_x + \sigma_z)] \ , \\ &\frac{\partial w}{\partial\zeta} = \varepsilon^{b}[\sigma_z - \nu(\sigma_x + \sigma_\theta)] \ , \qquad && \frac{\partial u}{\partial\zeta} + \varepsilon^{\lambda}\frac{\partial w}{\partial\xi} = \varepsilon^{b}\, 2(1+\nu)\sigma_{zx} \ ; \end{aligned}\right\} \quad (5)$$

Compatibility Equations,

$$\left.\begin{aligned} &\triangle\sigma_x + \frac{\varepsilon^{2\lambda}}{1+\nu}\frac{\partial^2\Omega}{\partial\xi^2} = 0 \ , \\ &\triangle\sigma_z + \frac{1}{1+\nu}\frac{\partial^2\Omega}{\partial\zeta^2} - 2g^2(\zeta)(\sigma_z - \sigma_\theta) = 0 \ , \\ &\triangle\sigma_\theta + \frac{g(\zeta)}{1+\nu}\frac{\partial\Omega}{\partial\zeta} + 2g^2(\zeta)(\sigma_z - \sigma_\theta) = 0 \ , \\ &\triangle\sigma_{zx} + \frac{\varepsilon^{\lambda}}{1+\nu}\frac{\partial^2\Omega}{\partial\zeta\partial\xi} - g^2(\zeta)\sigma_{zx} = 0 \ , \end{aligned}\right\} \quad (6)$$

where ν is Poisson's ratio and,

$$g(\zeta) = \frac{\varepsilon^{b}}{1+\varepsilon^{b}\zeta} \ , \qquad \Omega = \sigma_z + \sigma_\theta + \sigma_x \ ,$$

$$\triangle \equiv \frac{\partial^2}{\partial\zeta^2} + g(\zeta)\frac{\partial}{\partial\zeta} + \varepsilon^{2\lambda}\frac{\partial^2}{\partial\xi^2} \ .$$

To complete the formulation of the exact theory we obtain, from the specification of the applied external forces the following boundary conditions:

$$\left.\begin{aligned} &\sigma_{zx}(\xi,\pm 1;\varepsilon) = 0 \quad , \qquad \sigma_z(\xi,1;\varepsilon) = p_o(\xi;\varepsilon) \quad , \\ &\sigma_z(\xi,-1;\varepsilon) = p_1(\xi;\varepsilon) \ , \end{aligned}\right\} \tag{7a}$$

$$\left.\begin{aligned} &\sigma_x(0,\zeta;\varepsilon) = \bar{\sigma}(\zeta;\varepsilon) \ , \qquad \sigma_{zx}(0,\zeta;\varepsilon) = \bar{\tau}_1(\zeta;\varepsilon) \ , \\ &\sigma_x\left(\frac{L}{R(\frac{h}{R})^{a/b}},\zeta;\varepsilon\right) = \bar{\sigma}(\zeta;\varepsilon) \ , \qquad \sigma_{zx}\left(\frac{L}{R(\frac{h}{R})^{a/b}},\zeta;\varepsilon\right) = \bar{\tau}_2(\zeta;\varepsilon) \ .^{\dagger} \end{aligned}\right\} \tag{7b}$$

The functions $p_o(\xi;\varepsilon)$, $p_1(\xi;\varepsilon)$, $\bar{\sigma}(\zeta;\varepsilon)$, $\bar{\tau}_1(\zeta;\varepsilon)$ and $\bar{\tau}_2(\zeta;\varepsilon)$ are arbitrary to the extent that they form a system of forces in equilibrium and,

$$p_o \not\equiv p_I \ . \tag{8}$$

Eq. (8) implies that surface forces which only stretch or only compress the cylinder wall are not permitted. This restriction is not essential, it merely simplifies the ensuing calculations.

A complete mathematical formulation of the elastic problem for the rotationally symmetric deformations of a circular

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† $\bar{\tau}_1(\pm 1) = \bar{\tau}_2(\pm 1) = 0$ for continuity.

cylinder (formulation A) consists of the equilibrium equations (4), the stress-displacement relations (5) and the boundary conditions (7). A second and equivalent formulation (formulation B) consists of the equilibrium equations (4), the compatibility equations (6) and the boundary conditions (7). In the following section we employ formulation A; while in the boundary layer investigation in Sections 4 and 5 it is convenient to use formulation B.

Section 6 contains a summary of the shell theory.

3. The Interior Problems

To obtain in a systematic manner from the exact theory, (4) - (7), approximate "thin" and "thick" shell theories we adopt formulation A of the previous section and assume that each stress and displacement component, indicated by the generic symbol $\sigma(\xi,\zeta;\varepsilon)$, can be represented by a power series in ε:

$$\sigma(\xi,\zeta;\varepsilon) \sim \sum_{i=0}^{\infty} \sigma^{i}(\xi,\zeta)\varepsilon^{i} , \tag{9}$$

where $\sigma^{i} = 0$ if $i < 0$. The coefficients, $\sigma^{i}(\xi,\zeta)$, are referred to as the interior stress coefficients or the interior displacement coefficients, whichever the case may be. We assume that the prescribed external forces in (7) can be written as:

$$\begin{Bmatrix} p_o(\xi;\varepsilon) \\ p_I(\xi;\varepsilon) \end{Bmatrix} = \sum_{i=0}^{\infty} \begin{Bmatrix} p_o^i(\zeta) \\ p_I^i(\xi) \end{Bmatrix} \varepsilon^i \ ; \ \begin{Bmatrix} \bar{\sigma}(\zeta;\varepsilon) \\ \bar{\tau}_1(\zeta;\varepsilon) \\ \bar{\tau}_2(\zeta;\varepsilon) \end{Bmatrix} = \sum_{i=0}^{\infty} \begin{Bmatrix} \bar{\sigma}^i(\zeta) \\ \bar{\tau}_1^i(\zeta) \\ \bar{\tau}_2^i(\zeta) \end{Bmatrix} \varepsilon^i \ . \quad (10a,b)$$

Expansions of the form (9) for the stresses and displacements are substituted with (10) into (4), (5) and (7a). Equating of coefficients of the same power of ε in each of these equations results in systems of algebraic and differential equations and surface boundary conditions for the interior stress and displacement coefficients. From the coefficients of ε^n we obtain,

$$\frac{\partial \sigma_z^n}{\partial \zeta} + \frac{\partial \sigma_{zx}^{n-\lambda}}{\partial \xi} + \sum{}' (\sigma_z^i - \sigma_\theta^i) = 0 \ , \quad (11a)$$

$$\frac{\partial \sigma_{zx}^n}{\partial \zeta} + \frac{\partial \sigma_x^{n-\lambda}}{\partial \xi} + \sum{}' \tau_{zx}^i = 0 \ , \quad (11b)$$

$$\left.\begin{aligned} \frac{\partial u^n}{\partial \xi} &= \sigma_x^{n-a} - \nu(\sigma_\theta^{n-a} + \sigma_z^{n-a}) \ , \\ w^n &= \sigma_\theta^n - \nu(\sigma_x^n + \sigma_z^n) + [\sigma_\theta^{n-b} - \nu(\sigma_x^{n-b} + \sigma_z^{n-b})]\zeta \ , \end{aligned}\right\} \quad (12a,b)$$

$$\frac{\partial w^n}{\partial \zeta} = \sigma_z^{n-b} - \nu(\sigma_x^{n-b} + \sigma_\theta^{n-b}) \ , \qquad \frac{\partial u^n}{\partial \zeta} + \frac{\partial w^{n-\lambda}}{\partial \xi} = 2(1+\nu)\sigma_{zx}^{n-b} \ , \quad (12c,d)$$

$$\sigma_{zx}^n(\xi,\pm 1) = 0 \ , \quad \begin{Bmatrix} \sigma_z^n(\xi,1), \sigma_z^n(\xi,-1) \end{Bmatrix} = \begin{Bmatrix} p_o^n(\xi), p_I^n(\xi) \end{Bmatrix} \ . \quad (13a,b)$$

Here the binomial expansion of $(1+\varepsilon^b\zeta)^{-1}$ in $g(\zeta)$ has been used and,

$$\sum{}' A^i \equiv \sum_{i+(j+1)b=n} \sum (-1)^j \zeta^j A^i \ , \qquad i,j \geq 0 \ .$$

We shall now obtain from (11-13) appropriate differential equations and stress displacement relations for the interior stress and displacement coefficients. To begin, eqs. (12a,b) are solved for σ_x^n and σ_θ^n,

$$\left.\begin{aligned}
\sigma_x^n &= \frac{1}{1-\nu^2}\left(\frac{\partial u^{n+a}}{\partial\xi} + \nu w^n\right) + \frac{\nu}{1-\nu}\sigma_z^n - \frac{\nu}{1-\nu^2}\left[\sigma_\theta^{n-b} - \nu\left(\sigma_x^{n-b} + \sigma_z^{n-b}\right)\right]\zeta \ , \\
\sigma_\theta^n &= \frac{1}{1-\nu^2}\left(\nu\frac{\partial u^{n+a}}{\partial\xi} + w^n\right) + \frac{\nu}{1-\nu}\sigma_z^n - \frac{1}{1-\nu^2}\left[\sigma_\theta^{n-b} - \nu\left(\sigma_x^{n-b} + \sigma_z^{n-b}\right)\right]\zeta \ .
\end{aligned}\right\} \quad (14)$$

Then from (3) and (12d) with n = a we see that either,

$$\left.\begin{aligned}
&\frac{\partial u^a}{\partial\zeta} = -\frac{\partial w^{2a-b}}{\partial\xi} && \text{if } 2a - b \geq 0 \ , \ \text{(Case I)} \ , \\
\text{or,} \\
&u^a = U^a(\xi) && \text{if } 2a - b < 0 \ , \ \text{(Case II)} \ ,
\end{aligned}\right\} \quad (15)$$

where we have used, $\sigma^i = 0$ if $i < 0$. Case II leads to "membrane theories" and will not be discussed here. We will therefore restrict our attention to Case I. The special case, $2a-b = 0$ (Case I-A), is considered first.

Case I-A: In this case the coordinate scalings, (2a), become,†

$$\xi = \frac{x}{\sqrt{Rh}} \ , \qquad \zeta = \frac{r - R}{h} \ . \tag{16}$$

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† Johnson and Reissner[15] employed a scaling equivalent to (16) but arrived at it through *a priori* physical reasoning. We will show later that this is only scaling that gives a reasonable shell theory.

and, $\lambda = b - a = a$. It then follows from (11b) that,

$$\frac{\partial \sigma_{zx}^{n}}{\partial \zeta} = 0 \quad \text{if } n < a \,, \qquad \frac{\partial \sigma_{z}^{n}}{\partial \zeta} = 0 \quad \text{if } n < 2a \,.$$

Integrating these expressions and using the surface boundary conditions, (13), we find that,

$$\sigma_{zx}^{n} = 0 \quad \text{if } n < a \,, \qquad \sigma_{z}^{n} = p_{O}^{n} = p_{I}^{n} = 0 \quad \text{if } n < 2a \,, \tag{17a,b}$$

where (8) has been employed. We conclude from (17b) and (10a) that surface loads whose orders of magnitude are less than, ε^{2a}, and satisfy (8), cannot be prescribed.

It follows immediately from (12c) that,

$$w^{n}(\xi,\zeta) = W^{n}(\xi) \qquad \text{if } n < 2a \,, \tag{18}$$

and from (12a,c,d) that, within a rigid body motion,

$$u^{n}(\xi,\zeta) = 0 \qquad \text{if } n < a \,.$$

Integrating (12d) and using (17a) and (18) there results,

$$u^{n}(\xi,\zeta) = -\frac{dW^{n-a}(\xi)}{d\xi}\,\zeta + U^{n}(\xi) \qquad \text{if } a \leq n < 3a \,. \tag{19}$$

Substituting this result, (17b) and (18) into eqs. (14), the first 2a interior stress-displacement relations for the coefficients of σ_x and σ_θ can be written as,

$$\sigma_x^n(\xi,\zeta) = s_x^n(\xi)+\bar{s}_x^n(\xi)\zeta \;, \; \sigma_\theta^n(\xi,\zeta) = s_\theta^n(\xi)+\bar{s}_\theta^n(\xi)\zeta \quad \text{if } n < 2a \;, \quad (20)$$

where,

$$\left.\begin{aligned} s_x^n(\xi) &= \frac{1}{1-\nu^2}\left[\frac{dU^{n+a}(\xi)}{d\xi}+\nu W^n(\xi)\right], \quad s_\theta^n(\xi) = \frac{1}{1-\nu^2}\left[\nu\frac{dU^{n+a}(\xi)}{d\xi}+W^n(\xi)\right], \\ \bar{s}_x^n(\xi) &= -\frac{1}{1-\nu^2}\frac{d^2W^n(\xi)}{d\xi^2} \quad\quad\quad , \quad \bar{s}_\theta^n(\xi) = -\frac{\nu}{1-\nu^2}\frac{d^2W^n(\xi)}{d\xi^2} \;. \end{aligned}\right\} (21)^{\dagger}$$

σ_x^n and σ_θ^n are therefore known if $n < 2a$ when W^n and U^{n+a} are determined for $n < 2a$. We now obtain using (11) and (13) appropriate differential equations to determine these quantities as well as stress displacement relations for σ_{zx}^n and σ_z^n.

Substituting (20) into (11b) with n in the range, $a \leq n < 3a$, integrating with respect to ζ and employing (13a) we obtain with the aid of (17a):

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† The coefficients defined in (21) are related to the conventional stress resultants, $N_x(x)$ and $N_\theta(x)$, and couples, $M_x(x)$ and $M_\theta(x)$ of the classical shell theories.[1,16] If each of these quantities is expanded in a power series in ε then it is easy to show that,

$$s_x^n(\xi) = \frac{N_x^n(x)}{2Eh} \;, \qquad s_\theta^n(\xi) = \frac{N_\theta^n(x)}{2Eh} \;,$$

$$\bar{s}_x^n(\xi) = \frac{3M_x^n(x)}{2Eh^2} \;, \qquad \bar{s}_\theta^n(\xi) = \frac{3M_\theta^n(x)}{2Eh^2} \;.$$

$$\frac{dS_x^n}{d\xi} = 0 \qquad \text{if } n < 2a \ , \tag{22}$$

and,

$$\sigma_{zx}^n(\xi,\zeta) = \frac{1}{2}\,\frac{d\bar{S}_x^{n-a}}{d\xi}(1-\zeta^2) \qquad \text{if } a \leq n < 3a \ . \tag{23}$$

Eq. (22) yields one of the differential equations to determine each of the first 2a interior stress coefficients; while (23) gives the stress displacement relation for the first 2a non-vanishing σ_{zx}^n. To obtain the second differential equation and stress displacement relations for σ_z^n we consider (11a) with n in the range, $2a \leq n < 4a$. Substituting (20), (23) and (17b), integrating once with respect to ζ and inserting this result into (13b) we find that,

$$\frac{d^2\bar{S}_x^{n-2a}}{d\xi^2} - 3S_\theta^{n-2a} = 3(p_I^n - p_o^n) \qquad \text{if } 2a \leq n < 4a \ , \tag{24}$$

$$\sigma_z^n(\xi,\zeta) = \frac{1}{6}\,\frac{d^2\bar{S}_x^{n-2a}}{d\xi^2}\,\zeta^3 + \frac{1}{2}\bar{S}_\theta^{n-2a}\zeta^2 + \left(S_\theta^{n-2a} - \frac{1}{2}\,\frac{d^2\bar{S}_x^{n-2a}}{d\xi^2}\right)\zeta + (p_I^n + p_o^n) \tag{25}$$

$$\text{if } 2a \leq n < 4a \ .$$

It may now be assumed, with no loss of generality, that $p_o^{2a}(\xi) \equiv 0$ and $p_I^{2a}(\xi) \not\equiv 0$. This essentially determines the value of, a, and limits the surface forces to those whose orders of magnitude are not less than, $\varepsilon^{2a} = h/R$. Thus eqs. (22) and (24) are the differential equations necessary to determine the first 2a non-vanishing interior stress coefficients.

To summarize case I-A, we have obtained differential equations for the first 2a orders and stress displacement relations for each of the first 2a non-vanishing stress coefficients. All these results are deduced without reference to the boundary conditions (10b) along the edges. We shall now explicitly list the differential equations and stress displacement relations of the two lowest order interior problems. From (22) and (24) employing (20) and (21) the differential equations of the "zeroth order interior problem" are:

$$\left.\begin{aligned} \frac{dS_x^o}{d\xi} &= \frac{d^2U^a}{d\xi^2} + \nu\frac{dW^o}{d\xi} = 0 , \\ \frac{d^2\bar{S}_x^o}{d\xi^2} - 3S_\theta^o &= -\frac{1}{1-\nu^2}\left[\frac{d^4W^o}{d\xi^4} + 3W^o + 3\nu\frac{dU^a}{d\xi}\right] = 3(p_I^{2a} - p_o^{2a}) ; \end{aligned}\right\} \quad (26)$$

while the stress displacement relations (20), (23) and (25) may be written as,

$$\left.\begin{aligned} \sigma_x^o(\xi,\zeta) &= \frac{1}{1-\nu^2}\left[\frac{dU^a}{d\xi} + \nu W^o - \frac{d^2W^o}{d\xi^2}\zeta\right] , \\ \sigma_\theta^o(\xi,\zeta) &= \frac{1}{1-\nu^2}\left[\nu\frac{dU^a}{d\xi} + W^o - \nu\frac{d^2W^o}{d\xi^2}\zeta\right] , \end{aligned}\right\} \quad (27)$$

$$\left.\begin{aligned} \sigma_{zx}^a(\xi,\zeta) &= -\frac{1}{2(1-\nu^2)}\frac{d^3W^o}{d\xi^3}(1-\zeta^2) , \\ \sigma_z^{2a}(\xi,\zeta) &= \frac{1}{1-\nu^2}\left[-\frac{1}{6}\frac{d^4W^o}{d\xi^4}\zeta^3 - \frac{\nu}{2}\frac{d^2W^o}{d\xi^2}\zeta^2 + \left(\frac{1}{2}\frac{d^4W^o}{d\xi^4} + W^o + \nu\frac{dU^a}{d\xi}\right)\zeta\right] + (p_I^{2a} + p_o^{2a}) . \end{aligned}\right\} \quad (28)$$

Equations (26) and (27) are, in our notation, the differential equations and stress displacement relations of the classical theory of thin shells due to Love.[1,16] The relations (28) are not given in the classical theory. Similarly the differential equations and the stress displacement relations of the "first order interior problem" can be obtained from eqs. (20)-(25). They are identical with those given in (26)-(28) provided that, 1, is added to the superscript of each term. We call the result, eqs. (29). The resulting stresses, σ_x^1, σ_θ^1, σ_{zx}^{a+1} and σ_z^{2a+1}, which are determined from the solutions of (29) provide the first interior correction to the zeroth order interior problem consisting of the classical thin shell theory (26) and (27), and the additional relations (28).

To complete the formulation of the interior problems appropriate boundary conditions are derived in Sections 4 and 5 for the differential equations (26) and (29).

We continue in this fashion to obtain differential equations and stress displacement relations for the higher order interior problems. For example, for the 2a+1 interior problem we obtain,

$$\left.\begin{aligned}
w^{2a}(\xi,\zeta) &= \overset{2}{W}{}^{2a}(\xi)\zeta^2 + \overline{W}{}^{2a}(\xi)\zeta + W^{2a}(\xi) \;, \\
u^{3a}(\xi,\zeta) &= \overset{3}{U}{}^{3a}(\xi)\zeta^3 + \overset{2}{U}{}^{3a}(\xi)\zeta^2 + \overline{U}{}^{3a}(\xi)\zeta + U^{3a}(\xi) \;, \\
\sigma_x^{2a}(\xi,\zeta) &= \overset{3}{S}{}_x^{2a}(\xi)\zeta^3 + \overset{2}{S}{}_x^{2a}(\xi)\zeta^2 + \overline{S}{}_x^{2a}(\xi)\zeta + S_x^{2a}(\xi) \;, \\
\sigma_\theta^{2a}(\xi,\zeta) &= \overset{3}{S}{}_\theta^{2a}(\xi)\zeta^3 + \overset{3}{S}{}_\theta^{2a}(\xi)\zeta^2 + \overline{S}{}_\theta^{2a}(\xi)\zeta + S_\theta^{2a}(\xi) \;.
\end{aligned}\right\} \qquad (30)$$

The expressions for the coefficients of ζ^i, $i > 0$, in (30), which are functions of $W^o(\xi)$, $W^{2a}(\xi)$, $U^a(\xi)$, $U^{3a}(\xi)$, $p_o^{2a}(\xi)$ and $p_I^{2a}(\xi)$ will not be explicitly given.

This concludes the discussion of Case I-A. Case I-B for which, $2a-b > 0$, is analyzed in a similar manner. Some of the results are given in the appendix. We observe that the zeroth order interior stress coefficients can be determined only within an arbitrary linear function of ξ.

We define a "reasonable interior shell theory" as one whose differential equations, boundary conditions and stress displacement relations lead to uniquely determined interior stress and displacement coefficients of every order. In terms of this definition, Case I-B yields an unreasonable theory and is discarded. Thus we have shown that, within the limits of applicability of the expansion procedures presented in this paper, there is only one scaling of the coordinates (16) which can give a reasonable theory.

4. Formulation of the Boundary Layer Problem

It was shown in the previous section that the boundary values of, σ_x^i and σ_{zx}^i, are explicit functions of ζ [see eqs. (20) and (23)] while the applied edge stresses, $\overline{\sigma}(\zeta;\varepsilon)$ and $\overline{\tau}(\zeta;\varepsilon)$, are, within the limits of equilibrium and continuity, arbitrary functions of ζ. This indicates that, in general, <u>the interior expansions cannot satisfy the boundary conditions</u>

(7b) along the edge. Thus if the series (9) represent the solution of the exact theory, they do so in a region away from the edge, i.e. in the "interior" of the shell. The expansions therefore cannot hold uniformly up to the edge and, in fact, are not valid at the edge. The solutions of the exact theory rapidly deviate from the expansions (9) upon approaching the edge; the smaller ε is the more rapid the deviation. The region in which the rapid deviation occurs is called the boundary layer. In the limit as $\varepsilon \rightarrow 0$ the "width", δ, of this boundary layer $\rightarrow 0$. To obtain an approximation of the rapidly varying stress distribution near the edge we employ the technique[12,13,14] of expanding or "stretching" the variable ξ so that the resulting expansions uniformly represent the solution up to the edge. If we introduce the "boundary layer coordinate", η, defined by,†

$$\eta = \frac{\xi}{\varepsilon^{\lambda}} = \frac{\xi}{\left(\frac{h}{R}\right)^{1-a/b}} = \frac{x}{h} \quad , \tag{31}$$

where (2a) is used, then in the limit as $\varepsilon \rightarrow 0$ the width of the boundary layer which vanishes when expressed in the ξ variable, remains of non-vanishing length. In fact any fixed neighborhood of the edge in the ξ variable corresponds to an arbitrarily large neighborhood in the η variable provided ε is made sufficiently small.

The scaling and stretching of the axial coordinate introduced in (2a) and (31) yields a systematic method of studying

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† Here we are examining the boundary layer adjacent to the edge $\xi = x = 0$. An identical procedure applies to the edge $x = L$.

boundary layers of different widths. In fact if we define the order of magnitude of δ as the order of magnitude of the difference between the scaling and stretching lengths of the axial coordinate, x, then from (16) and (31) it follows (for Case I-A) that,

$$\delta = O[\sqrt{Rh}\ (1-\sqrt{\tfrac{h}{R}})] .$$

To investigate the boundary layer it is convenient to consider formulation B of Section 2 consisting of the equilibrium equations (4), the compatibility equations (6) and the boundary conditions (7). The coordinate η, eq. (31), is introduced into these equations and solutions are then sought in a form similar to (9). These solutions must satisfy the boundary conditions along the edges and surfaces of the cylinder. In addition they must "match", in a manner to be precisely formulated, with the interior solutions away from the edge. Appropriate boundary conditions for the differential equations of the interior problems are also obtained from this boundary layer investigation.

We define the "boundary layer stresses", $f(\eta,\zeta;\varepsilon)$, as,

$$\left.\begin{aligned} f_x(\eta,\zeta;\varepsilon) &= \frac{\tau_{xx}(x,r)}{E} , & f_\theta(\eta,\zeta;\varepsilon) &= \frac{\tau_{\theta\theta}(x,r)}{E} , \\ f_z(\eta,\zeta;\varepsilon) &= \frac{\tau_{rr}(x,r)}{E} , & f_{zx}(\eta,\zeta;\varepsilon) &= \frac{\tau_{rx}(x,r)}{E} , \end{aligned}\right\} \quad (32)$$

and,

$$\Lambda(\eta,\zeta;\varepsilon) = f_x + f_\theta + f_z .$$

It is assumed that each boundary layer stress component can be represented by a power series in ε of the form:

$$f(\eta,\zeta;\varepsilon) \sim \sum_{i=0}^{\infty} f^{i}(\eta,\zeta)\varepsilon^{i} , \tag{33}$$

where $f^{i} = 0$ if $i < 0$. The quantities $f^{i}(\eta,\zeta)$ are called the boundary layer stress coefficients. Substituting (31), (32) and expansions of the form (33) into (4) and (6) and equating coefficients of like powers of ε, we find from the coefficients of ε^{n}:

$$\left.\begin{aligned} \frac{\partial f_{z}^{n}}{\partial \zeta} + \frac{\partial f_{zx}^{n}}{\partial \eta} + \sum{}' \,(f_{z}^{i} - f_{\theta}^{i}) &= 0 , \\ \frac{\partial f_{zx}^{n}}{\partial \zeta} + \frac{\partial f_{x}^{n}}{\partial \eta} + \sum{}' \, f_{zx}^{i} &= 0 , \end{aligned}\right\} \tag{34}$$

$$\left.\begin{aligned} \nabla^{2} f_{x}^{n} + \frac{1}{1+\nu}\frac{\partial^{2}\Delta^{n}}{\partial \eta^{2}} + \sum{}' \frac{\partial f_{x}^{i}}{\partial \zeta} &= 0 , \\ \nabla^{2} f_{z}^{n} + \frac{1}{1+\nu}\frac{\partial^{2}\Delta^{n}}{\partial \zeta^{2}} + \sum{}' \frac{\partial f_{z}^{i}}{\partial \zeta} - 2\sum{}'' \,(f_{z}^{i} - f_{\theta}^{i}) &= 0 , \\ \nabla^{2} f_{\theta}^{n} + \sum{}' \left[\frac{1}{1+\nu}\frac{\partial \Delta^{i}}{\partial \zeta} + \frac{\partial f_{\theta}^{i}}{\partial \zeta}\right] + 2\sum{}''(f_{z}^{i} - f_{\theta}^{i}) &= 0 , \\ \nabla^{2} f_{zx}^{n} + \frac{1}{1+\nu}\frac{\partial^{2}\Delta^{n}}{\partial \zeta\,\partial \eta} + \sum{}' \frac{\partial f_{zx}^{i}}{\partial \zeta} - \sum{}'' \, f_{zx}^{i} &= 0 . \end{aligned}\right\} \tag{35}$$

Here, the binomial expansions of $g(\zeta)$ and $g^{2}(\zeta)$ are employed, $\sum{}'$ is defined in Section 3 and

$$\sum{}'' A^{i} \equiv \sum_{i+(j+2)b=n} \sum (-1)^{j}(j+1)\zeta^{j}A^{i}, \quad i,j \geq 0 \; ; \; \nabla^{2} \equiv \frac{\partial^{2}}{\partial \eta^{2}} + \frac{\partial^{2}}{\partial \zeta^{2}} .$$

Boundary conditions for the $f^n(\eta,\zeta)$ on $\zeta = \pm 1$ are obtained by assuming that $p_o^i(\xi)$ and $p_I^i(\xi)$ can be expanded in Taylor series of the form:

$$p^i(\xi) = \sum_{j=0}^{\infty} p_j^i \xi^j = \sum_{j=0}^{\infty} p_j^i \eta^j \varepsilon^{i\lambda} ,$$

where (31) is used and,

$$p_j^i = \begin{cases} \dfrac{1}{j!} \dfrac{d^j p^i(0)}{d\xi^j} , & j = 0,1,\ldots, \\ 0 , & j < 0 . \end{cases}$$

Employing this result and (10) we find from (31)-(33) the boundary conditions for $f^n(\eta,\zeta)$:

$$f_{zx}^n(\eta,\pm 1) = 0 , \quad \begin{Bmatrix} f_z^n(\eta,1) \\ f_z^n(\eta,-1) \end{Bmatrix} = \sum_{j=0}^{n/\lambda} \begin{Bmatrix} p_{oj}^{n-j\lambda} \\ p_{Ij}^{n-j\lambda} \end{Bmatrix} \eta^j , \tag{36a}$$

$$f_{zx}^n(0,\zeta) = \bar{\tau}_1^n(\zeta) , \qquad f_x^n(0,\zeta) = \bar{\sigma}^n(\zeta) . \tag{36b}$$

To complete the formulation of the boundary layer problem consisting of (34)-(36) we require that the boundary layer stress coefficients match the interior stress coefficients in the following manner. Assuming that, $\sigma^i(\xi,\zeta)$, are regular functions of ξ at $\xi = 0$, we expand them in a Taylor series in a neighborhood of the edge,

$$\sigma^i(\xi,\zeta) = \sum_{j=0}^{\infty} s_j^i(\zeta)\xi^j ,$$

where,

$$s_j^i(\zeta) = \begin{cases} \frac{1}{j!} \frac{\partial^j \sigma^i(0,\zeta)}{\xi^j} \, , & i \geq 0 \, , \\ 0 \, , & i < 0 \, . \end{cases}$$

It follows from this result, (9) and (31) that in a neighborhood of the edge,

$$\sigma(\xi,\zeta) \sim \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} s_j^i(\zeta) \eta^j \varepsilon^{i+j\lambda} = \sum_{n=0}^{\infty} \sigma^{*n}(\eta,\zeta)\varepsilon^n \, . \qquad (37a)$$

Here,

$$\sigma^{*n}(\eta,\zeta) = \sum_{j=0}^{n/\lambda} s_j^{n-j\lambda}(\zeta)\eta^j \, , \qquad (37b)$$

are the interior stress coefficients near the boundary as functions of η and ζ.

From the definitions (2b) and (32) it follows that,

$$\sigma(\xi,\zeta;\varepsilon) = f(\eta,\zeta;\varepsilon) \, .$$

Employing the representations (33) and (37a) we then find the matching conditions (or the asymptotic form) for the stress coefficients for large η. We write these as,

$$\lim_{\eta \to \infty} f^n(\eta,\zeta) = \sigma^{*n}(\eta,\zeta) \, , \qquad n = 0,1,\ldots \qquad (38)$$

Before examining the system (34)-(36) and (38) in detail a result is stated, without proof, concerning the boundary value problem for a function, $\phi(\eta,\zeta)$, defined on the semi-infinite strip, $\eta \geq 0$, $|\zeta| \leq 1$, and satisfying the biharmonic equation,

$$\nabla^4\phi = \frac{\partial^4\phi}{\partial\eta^4} + 2\frac{\partial^4\phi}{\partial\eta^2\partial\zeta^2} + \frac{\partial^4\phi}{\partial\zeta^4} = 0 ,$$

and the boundary conditions,[†]

$$\left.\begin{aligned} &\phi_{\zeta\zeta}(0,\zeta) = h_1(\zeta) , \quad \phi_{\eta\zeta}(0,\zeta) = h_2(\zeta) , \quad \phi_{\eta\eta}(x,\pm 1) = \phi_{\eta\zeta}(x,\pm 1) = 0 , \\ &\lim_{\eta\to\infty} (\phi_{\zeta\zeta}(\eta,\zeta) , \ \phi_{\eta\zeta}(\eta,\zeta)) = 0 . \end{aligned}\right\} \quad (39)$$

It is easily shown that if $\phi_{\zeta\zeta}$, $\phi_{\eta\zeta}$ and ϕ_ζ are uniformly continuous functions of ζ as $\eta \to \infty$, then in order that ϕ, ϕ_η and ϕ_ζ are single valued functions,

$$\int_{-1}^{1} h_1(\zeta)d\zeta = 0 , \quad \int_{-1}^{1} \zeta h_1(\zeta)d\zeta = 0 , \quad \int_{-1}^{1} h_2(\zeta)d\zeta = 0 . \qquad (40)$$

These results will frequently be referred to as the "integral relations".

5. Analysis of the Boundary Layer Problem

We consider Case I-A, b = 2a. Equations (34)-(36) yield, for n < 2a,

$$\frac{\partial f^n_z}{\partial\zeta} + \frac{\partial f^n_{zx}}{\partial\eta} = 0 , \qquad \frac{\partial f^n_{zx}}{\partial\zeta} + \frac{\partial f^n_x}{\partial\eta} = 0 , \qquad (41)$$

[†] Subscripts on $\phi(\eta,\zeta)$ indicate partial differentiation, e.g.,

$$\phi_{\eta\zeta} = \frac{\partial^2\phi}{\partial\eta\partial\zeta} .$$

$$\left.\begin{aligned} &\nabla^2 f_x^n + \frac{1}{1+\nu}\frac{\partial^2 \Lambda^n}{\partial \eta^2} = 0 , && \nabla^2 f_z^n + \frac{1}{1+\nu}\frac{\partial^2 \Lambda^n}{\partial \zeta^2} = 0 , \\ &\nabla^2 f_\theta^n = 0 \quad , && \nabla^2 f_{zx}^n + \frac{1}{1+\nu}\frac{\partial^2 \Lambda^n}{\partial \zeta \partial \eta} = 0 , \end{aligned}\right\} \quad (42)$$

$$\left.\begin{aligned} &f_{zx}^n(\eta,\pm 1) = f_z^n(\eta,\pm 1) = 0 , \\ &f_x^n(0,\zeta) = \bar{\sigma}^n(\zeta) , \qquad f_{zx}^n(0,\zeta) = \bar{\tau}_1^n(\zeta) . \end{aligned}\right\} \quad (43)$$

To apply the matching conditions (38) we must consider two separate cases for the range $n < 2a$: (i) $0 \leq n < a$, (ii) $a \leq n < 2a$. Note that if the external forces on the surfaces are of order ε^2 then $a = 1$ and (i) and (ii) become, $n = 0$ and $n = 1$, respectively. We find using the interior stress coefficients (17), (20), (21), (23) and (37b) that the matching conditions (38) are written as,

$$\lim_{\eta \to \infty} [f_x^n, f_z^n, f_\theta^n, f_{zx}^n] = \left\{\begin{aligned} &[S_x^n(0)+\bar{S}_x^n(0)\zeta \ , \ 0 \ , \ S_\theta^n(0)+\bar{S}_\theta^n(0)\zeta \ , \ 0] \text{ for (i)} \\ &[S_x^n(0)+\bar{S}_x^n(0)\zeta + \frac{d\bar{S}_x^{n-a}(0)}{d\xi}\zeta\eta \ , \ 0 \ , \ S_\theta^n(0)+\bar{S}_\theta^n(0)\zeta \\ &\quad + \frac{dS_\theta^{n-a}(0)}{d\xi}\eta + \frac{d\bar{S}_\theta^{n-a}(0)}{d\xi}\zeta\eta \ , \ \frac{1}{2}\frac{d\bar{S}_x^{n-a}(0)}{d\xi}(1-\zeta^2)] \\ &\text{for (ii)} , \end{aligned}\right\} \quad (44)$$

where, (22) is also used and an obvious notation is employed.

Solutions may be obtained to (41)-(44) by defining stress functions $\gamma^n(\eta,\zeta)$ on the semi-infinite strip, $\eta \geq 0$, $|\zeta| \leq 1$, such that,†

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† Subscripts on $\gamma(\eta,\zeta)$ indicate partial differentiation.

$$\left.\begin{array}{l} f_x^n = \gamma^n_{\zeta\zeta} , \qquad f_z^n = \gamma^n_{\eta\eta} , \qquad f_{zx}^n = -\gamma^n_{\eta\zeta} , \\ \text{and} \\ f_\theta^n = \nu(f_x^n + f_z^n) = \nu\nabla^2\gamma , \qquad 0 \le n < 2a . \end{array}\right\} \quad (45)$$

By direct substitution we see that (45) are formal solutions of (41) and (42) provided that $\gamma^n(\eta,\zeta)$ are biharmonic functions. To apply the integral relations (40) we define the reduced stress function, $\phi^n(\eta,\zeta)$, and the reduced boundary layer stress coefficients, $F^n(\eta,\zeta)$, as,

$$\left.\begin{array}{l} F_x^n = \phi^n_{\zeta\zeta} \equiv \gamma^n_{\zeta\zeta} - \lim\limits_{\eta\to\infty} f_x^n , \quad F_z^n = \phi^n_{\eta\eta} \equiv \gamma^n_{\eta\eta} - \lim\limits_{\eta\to\infty} f_z^n , \\ F_{zx}^n = \phi^n_{\eta\zeta} \equiv \gamma^n_{\eta\zeta} + \lim\limits_{\eta\to\infty} f_{zx}^n , \quad F_\theta^n = \nu\nabla^2\phi^n = \nu\nabla^2\gamma^n - \lim\limits_{\eta\to\infty} f_\theta^n . \end{array}\right\} \quad (46)$$

It then follows from (44), (45) and (43) that the ϕ^n satisfy,

$$\nabla^4\phi^n = 0 , \qquad 0 \le n < 2a , \qquad (47)$$

and assume the boundary values (40) where,

$$\left.\begin{array}{l} h_1(\zeta) = \bar{\sigma}^n(\zeta) - S_x^n(0) - \bar{S}_x^n(0)\zeta , \quad \text{if } 0 \le n < 2a , \\ h_2(\zeta) = \begin{cases} -\bar{\tau}_1^n(\zeta) , & \text{if } 0 \le n < a , \\ -\bar{\tau}_1^n(\zeta) + \frac{1}{2}\dfrac{d\bar{S}_x^{n-a}(0)}{d\xi}(1-\zeta^2) , & \text{if } a \le n < 2a . \end{cases} \end{array}\right\} \quad (48)$$

Substituting (48) into the integral relations (40) we find that,

$$\left.\begin{aligned} s_x^n(0) &= \frac{1}{2}\int_{-1}^{1}\bar{\sigma}^n(\zeta)d\zeta\ , && (49a)\\ \bar{s}_x^n(0) &= -\frac{1}{1-\nu^2}\frac{d^2W^n(0)}{d\xi^2} = \frac{3}{2}\int_{-1}^{1}\zeta\,\bar{\sigma}^n(\zeta)d\zeta\ , && (49b)\end{aligned}\right\}\ 0 \leq n < 2a$$

$$\left.\begin{aligned} &\int_{-1}^{1}\bar{\tau}_1^n(\zeta)d\zeta = 0\ , && 0 \leq n < a\ ,\\ &\frac{d\bar{S}_x^{n-a}(0)}{d\xi} = -\frac{1}{1-\nu^2}\frac{d^3W^{n-a}(0)}{d\xi^3} = \frac{3}{2}\int_{-1}^{1}\bar{\tau}_1^n(\zeta)d\zeta\ , && a \leq n < 2a.\end{aligned}\right\}\ (49c)$$

It follows from the first of (49c) that if an edge shear force is applied of order of magnitude less than ε^a, (see (10b)), then it must be self-equilibrating. In the following we restrict ourselves to those $\bar{\tau}^n(\zeta)$ and $\bar{\sigma}^n(\zeta)$ that are not self-equilibrating. Then without loss of generality, we assume that, $\bar{\sigma}^o(\zeta) \not\equiv 0$, $\bar{\tau}^a(\zeta) \not\equiv 0$, $\bar{\tau}^n(\zeta) \equiv 0$ if $n < a$. Equations (49a,b) then yield two of the boundary conditions for the first, 2a, interior problems, while the second of (49c) yields the remaining condition for the first, a, interior problems. The three boundary conditions for the differential equations of the <u>zeroth order interior</u> problem are, from (49),

$$\left.\begin{aligned} &s_x^o(0) = \frac{1}{2}\int_{-1}^{1}\bar{\sigma}^o(\zeta)d\zeta\ , \quad \frac{d^2W^o(0)}{d\xi^2} = -\frac{3(1-\nu^2)}{2}\int_{-1}^{1}\zeta\bar{\sigma}^o(\zeta)d\zeta\ ,\\ &\frac{d^3W^o(0)}{d\xi^3} = -\frac{3(1-\nu^2)}{2}\int_{-1}^{1}\bar{\tau}_1^a(\zeta)d\zeta\ .\end{aligned}\right\}\ (50)$$

Equations (50) are, in terms of the variables of this paper, the boundary conditions usually assumed in the classical thin shell theory of Love.[1,16] If a > 1 then the boundary conditions for the first order interior problem may also be obtained from (49). They are identical with those given in (50) provided that, 1, is added to the superscript of each term. We call these boundary conditions, eqs. (51). If a = 1, (49a,b) will give the first two boundary conditions for the first order interior problem. The remaining condition is obtained from a similar analysis of the boundary layer equations (41)-(43) and the matching conditions for $n = 2a$. These calculations are quite lengthy and are not exhibited. However they show that this boundary condition coincides with the third of (51). Thus (51) are the boundary conditions for the differential equations (29) of the first order interior problem for all values of, a.

It is important to observe that the proper boundary conditions for the interior problems are obtained by investigating the boundary layer. In addition, this investigation yields a sequence of boundary value problems, consisting of (47), (39) and (43) to systematically calculate the stresses in the neighborhood of the boundary. For convenience we list here the first two "boundary layer problems": the zeroth order boundary layer problem,

$$\left.\begin{aligned}
&\nabla^4\phi^0 = 0 \;, \\
&\phi^0_{\eta\zeta}(\eta,\pm 1) = \phi^0_{\eta\eta}(\eta,\pm 1) = 0 \;, \quad \lim_{\eta\to\infty}\,[\phi^0_{\eta\zeta}(\eta,\zeta),\phi^0_{\zeta\zeta}(\eta,\zeta)] = 0 \;, \\
&\phi^0_{\zeta\zeta}(0,\zeta) = \bar{\sigma}^0(\zeta) - \tfrac{1}{2}\int_{-1}^{1}\bar{\sigma}^0(\zeta)d\zeta - \tfrac{3}{2}\zeta\int_{-1}^{1}\zeta\bar{\sigma}^0(\zeta)d\zeta \;, \quad \phi^0_{\eta\zeta}(0,\zeta) = 0 \;,
\end{aligned}\right\} \quad (52)$$

the first order boundary layer problem,

$$\left.\begin{aligned}
&\nabla^4\phi^1 = 0 \;, \\
&\phi^1_{\eta\zeta}(\eta,\pm 1) = \phi^1_{\eta\eta}(\eta,\pm 1) = 0 \;, \quad \lim_{\eta\to\infty}\,[\phi^1_{\eta\zeta}(\eta,\zeta),\phi^1_{\zeta\zeta}(\eta,\zeta)] = 0 \;, \\
&\phi^1_{\zeta\zeta}(0,\zeta) = \bar{\sigma}^1(\zeta) - \tfrac{1}{2}\int_{-1}^{1}\bar{\sigma}^1(\zeta)d\zeta - \tfrac{3}{2}\zeta\int_{-1}^{1}\zeta\bar{\sigma}^1(\zeta)d\zeta \;, \\
&\phi^1_{\eta\zeta}(0,\zeta) = \begin{cases} -\bar{\tau}^1_1(\zeta) + \tfrac{3}{4}(1-\zeta^2)\displaystyle\int_{-1}^{1}\bar{\tau}^1_1(\zeta)d\zeta \;, & \text{if } a = 1 \;, \\ 0 \;, & \text{if } a > 1 \;. \end{cases}
\end{aligned}\right\} \quad (53)$$

6. A Summary of the Shell Theory

By the term, "thick shell theory" we imply a prescription for modifying the results of thin shell theory to obtain a more accurate approximation of the stresses in a shell considered as a <u>three dimensional elastic body</u>. We expect a thick shell theory to supply the dominant corrections to the results of thin shell theory over the entire cylindrical region.

In the preceeding sections we described a method of obtaining from the exact theory systems of differential

equations and stress displacement relations whose solutions systematically approximate the stresses in the cylinder. Two types of expansions in terms of ε, are employed: The first of these, the interior expansion, provides approximations of the stresses in the interior of the shell; while the second, the boundary layer expansion, yields a description of the stresses near the edge. Appropriate boundary conditions for the differential equations that result from the interior expansions are deduced from properties of the boundary layer expansions.

We now summarize the results of Sections 3, 4 and 5 to define a, "thick shell theory of order N". Let the stresses of the N^{th} order thick shell theory, represented by the generic symbol, $\sigma^{(N)}(\xi,\zeta,\varepsilon)$, be defined as,

$$\sigma^{(N)}(\xi;\zeta;\varepsilon) = \sum_{i=0}^{N}[\sigma^{i}(\xi,\zeta)+F^{i}(\xi/\varepsilon^{a},\zeta)]\varepsilon^{i}$$

where σ^{i} and F^{i} are, respectively, the interior and reduced boundary layer stress coefficients. Thus the thick shell theory of order zero for which $\sigma^{(o)} = \sigma^{o} + F^{o}$, is composed of the zeroth order interior problem, eqs. (26)-(28) and (50), and the zeroth order boundary layer problem, eqs. (46) and (52). Therefore this thick shell theory modifies the classical thin shell theory, eqs. (26), (27) and (50) by supplying a first approximation, valid in the interior, of the transverse shear and normal stresses (28), and a correction which results from (46) and (52), of the stresses in the neighborhood of the edge. In a similar manner the thick shell theory of order one for which

$\sigma^{(1)} = \sigma^0 + F^0 + (\sigma^1 + F^1)\varepsilon$, is obtained by combining the results of the zeroth order thick shell theory with the contributions of the first order interior problem (29) and (51) and the first order boundary layer problem (46) and (53). We can proceed in this fashion to obtain thick shell theories of any order.

Johnson and Reissner[15] proposed corrections to the results of thin shell theory which were obtained by an "interior expansion" procedure that is similar to, but less general than, ours. If we specialize our results to the particular problem treated in ref. 15 (for the isotropic case), it can be shown that their first correction coincides with that furnished by our <u>second order interior problem</u> (30) and is therefore $O(\varepsilon^2)$. Since the boundary layer is not analyzed in ref. 15, the lower order corrections resulting from the zeroth and first order boundary layer problems and the first order interior problem are omitted.

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Appendix

Case I-B, $2a - b > 0$: We can show from (11)-(13) that,

$$w^n(\xi,\zeta) = W^n(\xi) = c_1^n\xi + c_2^n ,$$

$$u^{\lambda+n}(\xi,\zeta) = - c_1^n\zeta, \quad \text{if } n < 2a-b, \text{ or } \lambda+n < a ,$$

$$u^a(\xi,\zeta) = - \frac{dW^{2a-b}(\xi)}{d\xi}\zeta + U^a(\xi) , \quad \text{if } n = a ,$$

$$\sigma_x^o(\xi,\zeta) = T_x^o(\xi)+\overline{T}_x^o(\xi)\zeta + \frac{\nu}{1-\nu}\sigma_z^o , \quad \sigma_\theta^o(\xi,\zeta) = T_\theta^o(\xi)+\overline{T}_\theta^o(\xi)\zeta + \frac{\nu}{1-\nu}\sigma_z^o ,$$

where,

$$T_x^o(\xi) = \frac{1}{1-\nu^2}\left[\frac{dU^a(\xi)}{d\xi}+\nu c_1^o\xi+\nu c_2^o\right] , \quad T_\theta^o(\xi) = \frac{1}{1-\nu^2}\left[\nu\frac{dU^a(\xi)}{d\xi}+c_1^o\xi+c_2^o\right] ,$$

$$\overline{T}_x^o(\xi) = - \frac{1}{1-\nu^2}\frac{d^2W^{2a-b}(\xi)}{d\xi^2} , \quad \overline{T}_\theta^o(\xi) = - \frac{\nu}{1-\nu^2}\frac{d^2W^{2a-b}(\xi)}{d\xi^2} .$$

By proceeding in a fashion similar to that used in Case I-A, we can deduce, assuming $p^n = 0$ if $n < 2\lambda$, that,

$$\sigma_{zx}^n = 0 \qquad \text{if } n < \lambda ,$$

$$\sigma_z^n = 0 \qquad \text{if } n < 2\lambda ,$$

$$\sigma_{zx}^\lambda = \frac{1}{2}\frac{d\overline{T}_x^o}{d\xi}(1-\zeta^2) ,$$

$$\sigma_z^{2\lambda} = - \frac{1}{2}\frac{d^2\overline{T}_x^o}{d\xi^2}\left(\zeta - \frac{\zeta^3}{3}\right)+p_o^{2\lambda}+p_I^{2\lambda} ,$$

$$\frac{dT_x^o}{d\xi} = \frac{1}{1-\nu^2}\left[\frac{d^2U^a}{d\xi^2}+\nu c_1^o\right] = 0 ,$$

$$- \frac{d^2\overline{T}_x^o}{d\xi^2} = \frac{1}{1-\nu^2}\frac{d^4W^{2a-b}}{d\xi^4} = 3(p_o^{2\lambda}-p_I^{2\lambda}) ,$$

where c_1^n and c_2^n are arbitrary constants. The above equations form the stress displacement relations and the differential equations for the zeroth order interior problem of Case I-B.

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